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RESONANCE WAVES IN A MODEL OF A TWO-LAYERED LIQUID

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With allowance for surface interaction between phases, the behavior of long-wave perturbations at the interface between two layers of dissimilar liquids, which form resonance triplets described by a pseudodifferential equation, is studied.

1. We consider a system consisting of two ideal immiscible liquids in a gravity field. The heavy liquid of density ρ_0 occupies the lower part of the half-space ($z < 0$). Above, there is a thin layer of a lighter liquid of density ρ_1, bounded from above by a rigid horizontal plane. We introduce the following notation: h_0 is the thickness of the upper layer in an unperturbed system and σ is the coefficient of surface interaction between the phases.

In the case of very thin layers, the problem for long-wave interfacial perturbations reduces to an analysis of solutions of the following evolution equation obtained in [1]:

$$u_t + (u + u^2 - \alpha L u - \beta u_{xx})_x = 0. \tag{1.1}$$

Here u is the displacement of the interface; the subscripts t and x denote differentiation with respect to t and x, respectively; L is a linear, symmetrical pseudodifferential operator whose action in the k-space reduces down to multiplication of the corresponding Fourier-harmonic by $|k|$.

In derivation of Eq. (1.1), the conditions $\alpha = \rho_2/(2\rho_1)$ and $\beta = \sigma/[g(\rho_2 - \rho_1)h_0^2]$ ($\beta \gg \alpha$) were assumed to be satisfied.

Equation (1.1) is a long-wave evolution equation written with allowance for weak nonlinearity and dispersion for a wave propagating in one direction. With $\alpha = 0$, this equation reduces to the well-known Korteweg–de-Vries equation, and with $\beta = 0$ to the Benjamin–Ono equation [2]. This equation, in particular, is a good model for studying internal waves in a stratified liquid. The forth term in (1.1) takes into account the nonlocal relation between interface and pressure perturbations, and the last term models the dispersion caused by the forces due to surface interaction between the phases.

Equation (1.1) was also obtained by O. S. Ryzhov [3]; it was used to study boundary-layer perturbations.

With a special substitution, Eq. (1.1) can be rewritten in the form [4]

$$H_t + (H^2 - LH - H_{xx})_x = 0. \tag{1.2}$$

Steady-state traveling periodic and soliton solutions of Eq. (1.2) were numerically constructed in [4]. Solutions of a small but finite amplitude for a given wavenumber were represented in the form of a series in a small parameter. It was shown [4] that these solutions are regular for all wavenumbers k except for a vicinity of a singular point $k_* = 1/3$ because, in this case, the phase velocities of linear perturbations for the first and second harmonics coincide, and conditions for their resonance are satisfied [5].

The purpose of the present work is to study the interaction between perturbation triplets that satisfy spatial-synchronism conditions.

With the neglected nonlinear term in (1.2), the phase velocity of periodic infinitesimally small (linear) perturbations is a function of the wavenumber $k = 2\pi/\lambda$ (λ is the wavelength):

$$c = k^2 - |k|. \tag{1.3}$$

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It follows from (1.3) that Eq. (1.2) is written in a coordinate system moving with the phase velocity of infinitely long ($k = 0$) linear waves. From (1.2) and (1.3), it also follows that the resonance condition is valid for any three harmonics with infinitesimal amplitudes, provided that their wavenumbers satisfy the relations

$$k_1^0 + k_2^0 = k_3^0, \quad k_1^0 c_1^0 + k_2^0 c_2^0 = k_3^0 c_3^0, \quad c_i^0 = k_i^{02} - |k_i^0|. \tag{1.4}$$

2. We consider evolution of three harmonics that form a resonance triplet. Assuming the harmonic amplitudes to be small but finite, we seek the solution of (1.2) in the form of a series

$$H = \varepsilon H_1 + \varepsilon^2 H_2 + \dots. \tag{2.1}$$

We introduce a set of fast and slow times

$$t_n = t\varepsilon^n \quad (n = 0, 1, 2, \dots). \tag{2.2}$$

Substituting (2.1) into (1.2), with allowance made for (2.2), we obtain an infinite system of linear equations. In this system, the following equation corresponds to the first order in ε:

$$\frac{\partial H_1}{\partial t_0} - \frac{\partial}{\partial x}\Big(LH_1 + \frac{\partial^2 H_1}{\partial x^2}\Big) = 0. \tag{2.3}$$

The conditions of perfect synchronism for three harmonics with infinitesimal amplitudes require relations (1.4) to be valid. The solution of system (1.4) has the form

$$k_3^0 = 2/3, \qquad k_1^0 + k_2^0 = 2/3.$$

We consider perturbations in the form of some combination of three harmonics with small but finite amplitudes whose wavenumbers k_1, k_2, and k_3 are quite defined: each of them lies in a vicinity of the corresponding wavenumber k_i^0. In this situation, the condition of synchronism between wavenumbers

$$k_1 + k_2 = k_3 \tag{2.4}$$

is exactly fulfilled for these harmonics. For frequencies, these conditions are violated because

$$k_i = k_i^0 + \Delta_i, \qquad \Delta_i \ll k_i^0. \tag{2.5}$$

Apparently, for relation (2.4) to be fulfilled, the deviations of the wavenumbers Δ_i have to satisfy the relation

$$\Delta_1 + \Delta_2 = \Delta_3. \tag{2.6}$$

We seek the solution of Eq. (2.3) in the form

$$H_1 = A \exp[ik_1(x - c_1^0 t)] + B \exp[ik_2(x - c_2^0 t)] + D \exp[ik_3(x - c_3^0 t)] + \text{c.c.} \tag{2.7}$$

Here c.c. is a complex conjugate expression, k_i are the wavenumbers for which relations (2.4) and (2.5) are fulfilled, and c_i^0 satisfy relations (1.4). Since the amplitudes of harmonics A, B, and D are now small but finite, each of them are some function of time due to nonlinear interactions. We construct those solutions where these amplitudes are functions of times slower than $t_0 \equiv t$. For this, the condition

$$O(\Delta_i) \leqslant O(\varepsilon) \tag{2.8}$$

is to be satisfied. Indeed, substituting (2.6) into (2.3) and equating the coefficients at identical exponents to zero, for the function A, for instance, we obtain the equation

$$\Big[\frac{\partial A}{\partial t_0} - ik_1(c_1^0 + |k_1| - k_1^2)A\Big] \exp[ik_1(x - c_1^0 t_0)] = 0. \tag{2.9}$$

It follows from (1.3) and (2.9) that, for the condition $\partial A/\partial t_0 = 0$ to be fulfilled, the value of Δ_1 should not be greater, in the order of smallness, than ε. In what follows, we assume that these quantities are of the same order of smallness, i.e., relation (2.8) transforms into the equality. Then, the term $ik_1(c_1^0 + |k_1| - k_1^2)A$ in (2.9), which is proportional to A, should be transferred into the second-order equation. Analogous manipulations may also be done with the other harmonics. In this case, the equation

$$\underline{\frac{\partial H_1}{\partial t_1}} + \frac{\partial H_2}{\partial t_0} + \underline{\frac{\partial (H_1^2)}{\partial x}} - \frac{\partial}{\partial x}\Big(LH_2 + \frac{\partial^2 H_2}{\partial x^2}\Big)$$

$$+ \underline{\text{terms of order } \Delta/\varepsilon \text{ from the } \varepsilon\text{-approximation}} = 0 \tag{2.10}$$

corresponds to the next order in ε. The last term in (2.10) represents terms from the first approximation of (2.3). In Eq. (2.10), secular terms are underlined (for H_1^2, only part of all terms are secular). For a restricted solution H_2 to exist, it is necessary that the secular terms be equal to zero. This requirement leads to the following system of equations for the harmonic amplitudes A, B, and D:

$$\frac{dA}{dt_1} + ik_1^0\frac{\Delta_1}{\varepsilon}(2k_1^0 - 1)A + 2i(k_3 - k_2)\bar{B}D\exp\left[-i(k_3c_3^0 - k_2c_2^0 - k_1c_1^0)t_0\right] = 0,$$

$$\frac{dB}{dt_1} + ik_2^0\frac{\Delta_2}{\varepsilon}(2k_2^0 - 1)B + 2i(k_3 - k_1)\bar{A}D\exp\left[-i(k_3c_3^0 - k_1c_1^0 - k_2c_2^0)t_0\right] = 0, \qquad (2.11)$$

$$\frac{dD}{dt_1} + ik_3^0\frac{\Delta_3}{\varepsilon}(2k_3^0 - 1)D + 2i(k_1 + k_2)AB\exp\left[i(k_3c_3^0 - k_1c_1^0 - k_2c_2^0)t_0\right] = 0.$$

From here on, the bar denotes the complex conjugation operation.

Under the assumption that the small parameter ε in (2.1) is, for instance, deviation of the first-harmonic wavenumber Δ_1 from k_1^0 (i.e., $\varepsilon \equiv \Delta_1$), and with relation (2.6) between the deviations of the wavenumbers from linear resonance harmonics taken into account, after the substitution

$$A \to A\exp(i\gamma t_1/2), \quad B \to B\exp(i\gamma t_1/2), \quad D \to D$$

we may write system (2.11) in the form

$$\frac{dA}{dt_1} + i\Big[\frac{\gamma}{2} + k_1^0(2k_1^0 - 1)\Big]A + 2ik_1^0\bar{B}D = 0, \quad \frac{dB}{dt_1} + i\Big[\frac{\gamma}{2} + k_2^0\Delta(2k_2^0 - 1)\Big]B + 2ik_2^0\bar{A}D = 0,$$

$$\frac{dD}{dt_1} + i\,\frac{2(1+\Delta)}{9}\,D + \frac{4}{3}\,iAB = 0. \qquad (2.12)$$

Here $\Delta = \Delta_2/\Delta_1$ and $\gamma = k_1^{02}(1+\Delta) - k_1^0(1+\Delta/3) + 2/9$.

To examine the solutions of system (2.12), we represent the complex functions A, B, and D as

$$A = A_1\exp(i\varphi_A), \quad B = B_1\exp(i\varphi_B), \quad D = D_1\exp(i\psi),$$

where A_1, φ_A, B_1, φ_B, D_1, and ψ are real-valued functions.

Separating real and imaginary parts in (2.12), we obtain a system of six equations for these real-valued functions:

$$\frac{dA_1}{dt_1} = 2k_1^0B_1D_1\sin\chi, \quad \frac{dB_1}{dt_1} = 2k_2^0A_1D_1\sin\chi, \quad \frac{dD_1}{dt_1} = -\frac{4}{3}\,A_1B_1\sin\chi,$$

$$\frac{d\varphi_A}{dt_1} = -\Big[\frac{\gamma}{2} + k_1^0(2k_1^0 - 1)\Big] - 2k_1^0\,\frac{B_1D_1}{A_1}\,\cos\chi,$$

$$\frac{d\varphi_B}{dt_1} = -\Big[\frac{\gamma}{2} + k_2^0(2k_2^0 - 1)\Delta\Big] - 2k_2^0\,\frac{A_1D_1}{B_1}\,\cos\chi, \qquad (2.13)$$

$$\frac{d\psi}{dt_1} = -\frac{2}{9}(1+\Delta) - \frac{4}{3}\,\frac{A_1B_1}{D_1}\,\cos\chi.$$

Here $\chi = \psi - \varphi_A - \varphi_B$.

The first three equations of system (2.13) yield the integral

$$A_1^2 + B_1^2 + D_1^2 = C_{00}, \qquad (2.14)$$

and from the first two equations, we obtain the second integral of motion,

$$k_1^0B_1^2 - k_2^0A_1^2 = k_1^0C_1. \qquad (2.15)$$

With (2.14) and (2.15), from (2.13) we obtain the truncated system

$$\frac{dA_1}{dt_1} = 2k_1^0B_1D_1\sin\chi, \quad \frac{d\chi}{dt_1} = \gamma_1 + \Big[2k_1^0\,\frac{B_1D_1}{A_1} + 2k_2^0\,\frac{A_1D_1}{B_1} - \frac{4}{3}\,\frac{A_1B_1}{D_1}\Big]\cos\chi, \qquad (2.16)$$

where $\gamma_1 = \gamma + k_1^0(2k_1^0 - 1) + \Delta k_2^0(2k_2^0 - 1) - 2(1+\Delta)/9$.

The solutions of system (2.14)–(2.16) by no means provide exhaustive information about the evolution of solution (2.7) (since, with these solutions, only information about the behavior of some combination of the phases χ, but not about the behavior of the phases ψ, φ_A, and φ_B themselves, can be obtained). To withdraw the latter information, one has to solve system (2.13). System (2.14)–(2.16) can easily be solved in quadratures, and, analyzing the structure of phase curves of the solutions of this system, for instance, in the plane (A_1, χ), one can also gain rather comprehensive general information about solution (2.7).

According to (2.16), the phase curves are given by the equation

$$-\frac{d\cos\chi}{dA_1} = \frac{\gamma_1}{2k_1^0 B_1 D_1} + \Big(\frac{1}{A_1} + a\,\frac{A_1}{B_1^2} - \frac{2}{3k_1^0}\,\frac{A_1}{D_1^2}\Big)\cos\chi,$$

whose solutions have the form

$$\cos\chi = (C - a_1 A_1^2)/[2(C_1 + aA_1^2)^{1/2}(C_{00} - C_1 - a_2 A_1^2)^{1/2} A_1]. \tag{2.17}$$

Here $a = k_2^0/k_1^0$, $a_2 = 1 + a$, and C is the constant of integration for a particular trajectory defined by some initial conditions:

$$C = 2A_1^0(C_1 + aA_1^{02})^{1/2}(C_{00} - C_1 - a_2 A_1^{02})^{1/2}\cos\chi_0 + a_1 A_1^{02}$$

(A_1^0 and χ_0 are the coordinates of the initial point that defines the trajectory).

In spite of the fact that the phase trajectories are explicitly given by simple formulas (2.14), (2.15), and (2.17), even an analysis of the phase portraits of the solutions is hampered because the parameters involved are too numerous. Representation of the characteristic solutions of system (2.13) and corresponding solutions (2.7) is also a difficult problem. However, the analysis is facilitated by some special structural features. For instance, in the case of $k_1^0 = k_2^0 = 1/3$ and $\Delta = 1$, the problem degenerates, and after some renormalizations ($A + B \to A$), we obtain a system for a two-wave resonance [5]. If $\Delta \neq 1$, then two harmonics out of three have close wavenumbers.

System (2.13) is invariant with respect to the transform

$$A_1 \to -A_1, \quad B_1 \to -B_1, \quad D_1 \to -D_1, \quad \chi \to \pi - \chi, \quad t_1 \to -t_1.$$

In addition, the transform

$$A_1 \to A_1, \quad B_1 \to -B_1, \quad D_1 \to -D_1$$

is valid, as well as the transform

$$A_1 \to -A_1, \quad B_1 \to -B_1, \quad D_1 \to D_1.$$

Since the phase space of the solutions of system (2.13) has a period 2π along the variable χ, we may restrict ourselves, for instance, to consideration of the region $0 < \chi < 2\pi$, $A_1 > 0$ in the phase plane (A_1, χ).

It follows from (2.16) that the phase space may have stationary points for which the following relations hold:

$$\chi = 0, \pi, \qquad \gamma_1 = \mp\Big(2k_1^0\,\frac{B_1 D_1}{A_1} + 2k_2^0\,\frac{A_1 D_1}{B_1} - \frac{4}{3}\,\frac{A_1 B_1}{D_1}\Big). \tag{2.18}$$

The solutions H_1 corresponding to these points represent a nonlinear superposition of three steady-state traveling waves, each having its own phase velocity and, generally speaking, incomparable wavenumbers. In addition, the phase velocity of each wave differs from the corresponding value c_i^0. One of the reasons for this difference consists in the fact that, as it follows from (2.13), each of the phases ψ, φ_A, and φ_B changes with its own constant rate in spite of the constancy of χ. The solutions H_1 that correspond to nontrivial phase curves (2.17) have even a more complex structure.

Apart from stationary points (2.18), important structural elements of the phase space are separatrices on which the amplitude of a harmonic passes through zero. Formulas for the projections of these separatrices onto the phase plane (A_1, χ) are given below.

1. Separatrix on which $A_1^0 = 0$:

$$\cos\chi = -a_1 A_1/[2(C_1 + aA_1^2)^{1/2}(C_{00} - C_1 - a_2 A_1^2)^{1/2}]. \tag{2.19}$$

2. Separatrix on which $B_1^0 = 0$:

$$\cos\chi = -a_1(C_1 + aA_1^2)^{1/2}/[2a(C_{00} - C_1 - a_2 A_1^2)^{1/2} A_1]. \tag{2.20}$$

A comparison between formulas (2.19) and (2.20) performed with due regard for the conservation integrals (2.14) and (2.16) (from which it follows that $C_1 = B_1^{02}$ in the first case and $C_1 = -A_1^{02}/a$ in the second case) shows that these formulas define one and the same curve only if $C_1 = 0$. In all other cases, one of these harmonics is necessarily a non-zero one.

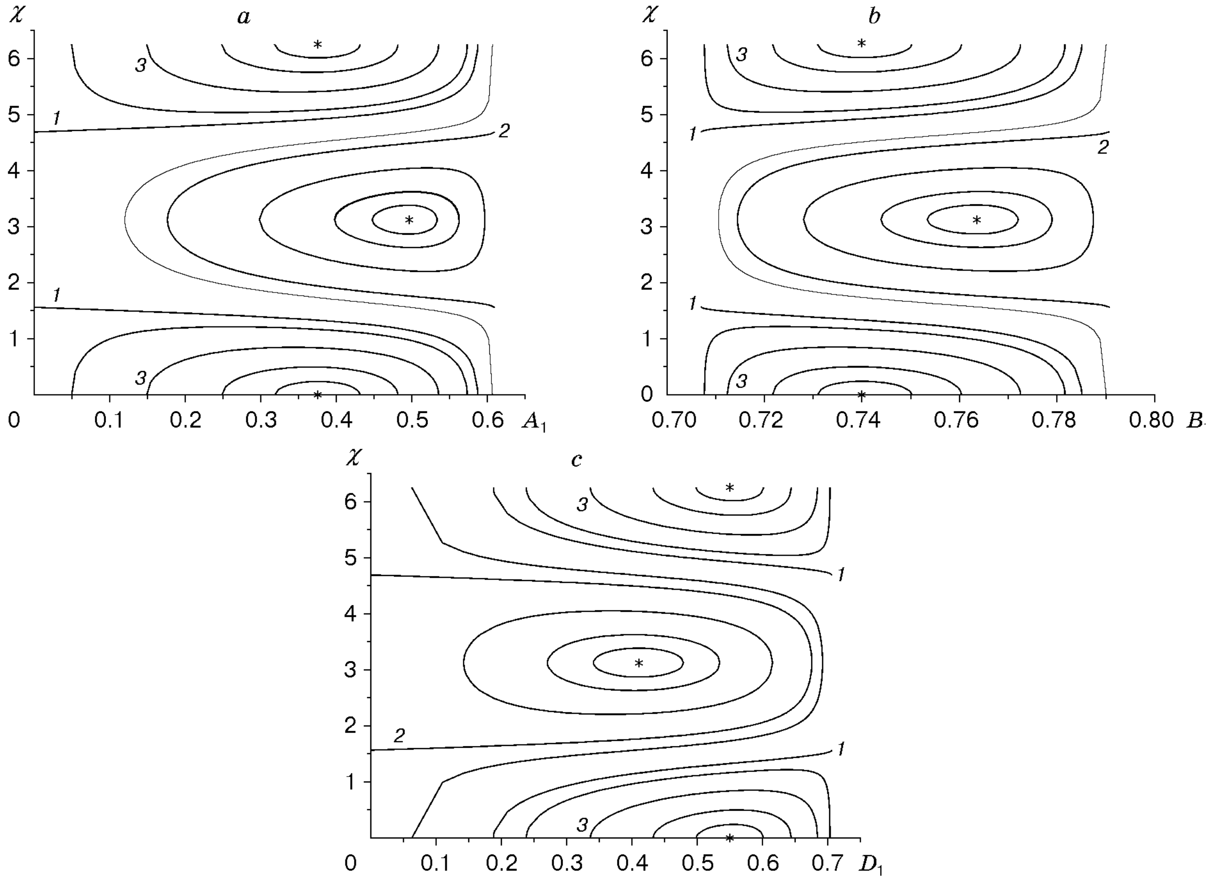

Fig. 1. Projections of phase trajectories onto the planes (A_1, χ) (a), (B_1, χ) (b), and (D_1, χ) (c) ($C_{00} = 1$, $C_1 = 0.5$, $k_1^0 = 0.5$, and $k_2^0 = 0.166\,667$): 1) separatrix A_1^0; 2) separatrix D_1^0; 3) closed trajectories.

3. Separatrix on which $D_1^0 = 0$:

$$\cos\chi = a_1(C_{00} - C_1 - a_2 A_1^2)^{1/2}/[2a_2(C_1 + aA_1^2)^{1/2}A_1].$$

It follows from an analysis of the phase curves that the behavior of an affix that moves in the phase space along the corresponding separatrix trajectory is rather simple: in a finite time, it leaves the point in which the harmonics that defines it equals zero and, also in a finite time, reaches another similar point. On passing through zero, this harmonic [for separatrix (2.19), for instance, this harmonic is A_1] changes its sign. At these moments, two other harmonics reach their extreme values for the given solution. On reaching these points, the affix projection onto the plane with the values of the corresponding harmonic and the values of χ are plotted along the axes [in the above-indicated case, these planes are (B_1, χ) and (D_1, χ)] starts moving in the opposite direction.

A special case is observed for $C_1 = 0$. In this case, as was noted above, formulas (2.19) and (2.20) define one curve. In this curve, the harmonics A_1 and B_1 reach their ultimate values as $t_1 \to \pm\infty$, i.e., to this separatrix, a solution of system (2.13) corresponds; the contribution of the harmonics A_1 and B_1 to this solution is an envelope soliton. Thus, over long times, the solution of (2.13) for these parameters is an almost periodic steady-state traveling wave with the wavenumber k_3 and amplitude D_1^0. Although the behavior of phase trajectories provides rather extensive information about the corresponding solution (2.7), in order to completely describe it, one has to consider system (2.13). This system was numerically solved by the fifth-order Runge–Kutta method with an automatic choice of the step and control of computation accuracy.

Figure 1 shows the phase portrait of the solutions of system (2.16) for a typical case of $C_{00} = 1$ and $C_1 = 0.5$. The projections of phase trajectories onto the planes (A_1, χ), (B_1, χ), and (D_1, χ) for $k_1^0 = 1/2$, $k_2^0 = 1/6$, and $\Delta = 1$ are shown. It is seen that this region contains two stationary points (with due regard for the periodicity of the phase space in χ). In Fig. 1, these singular points are shown as asterisks. Both points are of the "center" type, and their coordinates A_1, B_1, and D_1, and χ are 0.38, 0.74, and 0.5545, and 0 and 0.497, 0.763, and 0.413, and π, respectively.

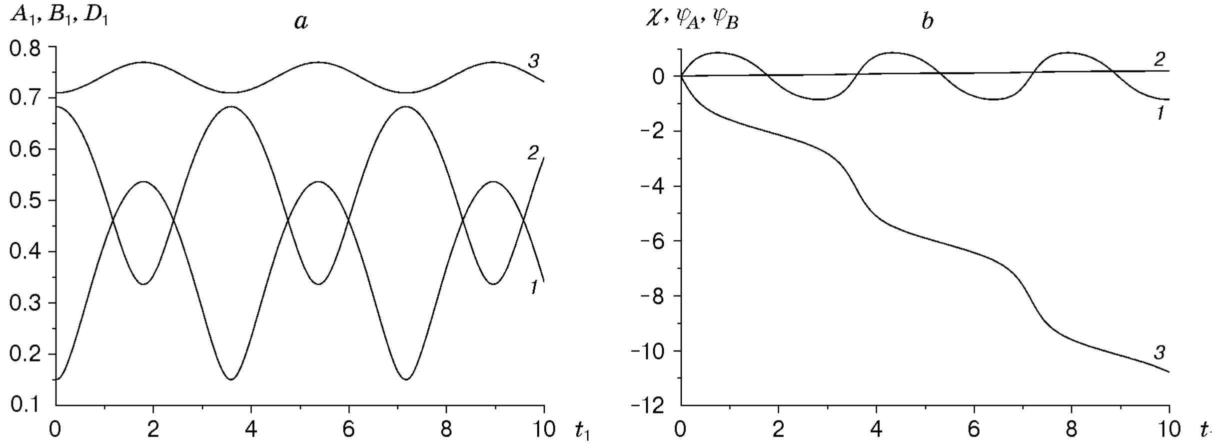

Fig. 2. Amplitudes of harmonics (a) and their phases (b) versus time ($C_{00} = 1$, $C_1 = 0.5$, $k_1^0 = 0.5$, $k_2^0 = 0.166\,667$, and $\Delta = 1$): (a) curves 1–3 refer to A_1, B_1, and D_1 harmonics; (b) curves 1–3 refer to χ, φ_B, and φ_A phases.

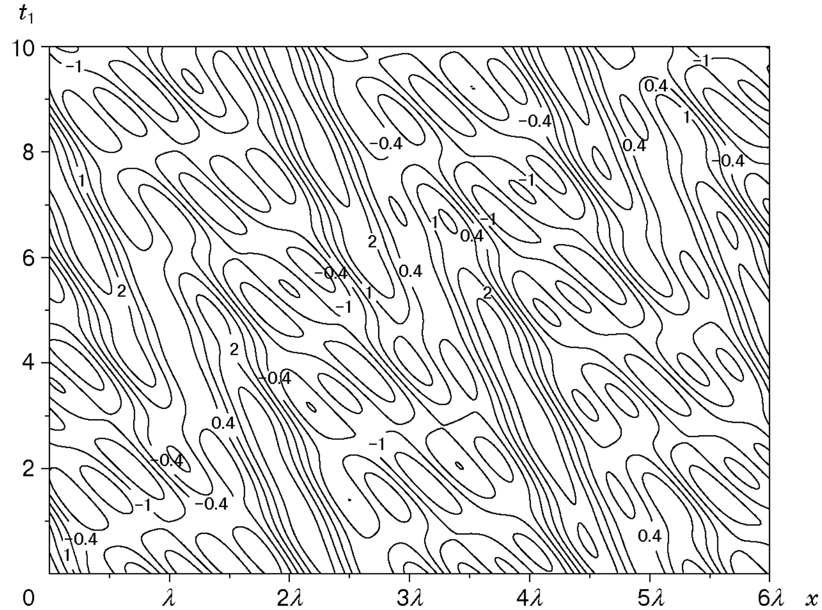

Fig. 3. Isolines of the solution H_1, for phase curve 3 in Fig. 1 ($C_{00} = 1$, $C_1 = 0.5$, $k_1^0 = 0.5$, $k_2^0 = 0.166\,667$, $\Delta = 1$, and $\Delta_1 = 0.1$).

Conventionally, two types of trajectories may be distinguished. The first type includes trajectories closed around corresponding centers. Along these trajectories, the quantities χ, A_1, B_1, and D_1 exert periodic fluctuations. For small deviations from the corresponding stationary point, the fluctuations of these functions are almost harmonic ones with a frequency proportional to the "squared amplitude of deviation" from the given point. As the trajectories depart from the stationary point, the fluctuations acquire a more evolved character. In the limiting case, these closed phase curves transform into corresponding separatrix trajectories that separate them out from open trajectories. The second-type trajectories, as projected onto an arbitrary plane drawn through the χ axis, are open curves that lie between the separatrices.

Figure 2 shows the time evolution of the solution of system (2.16) for the phase trajectory shown by curves 3 in Fig. 1. It follows from Fig. 2 that the amplitudes of harmonics vary appreciably (Fig. 2a), the phase difference χ always remains restricted (curve 1 in Fig. 2b), and the phase of the harmonic $A_1 \to \varphi_A$, fluctuating, decreases indefinitely (curve 3 in Fig. 2b). As a result, the form of the corresponding solution H_1 (Fig. 3) bears no resemblance to a traveling wave. In the plane (x, t_1), the isolines have no definite slope and display no periodicity, although, here, the shown interval in the x-direction is as long as six wavelengths of the harmonic D_1 (the quantity $\lambda = 2\pi/k_3^0$, where $k_3^0 = 2/3$, is used here as a spatial length scale).

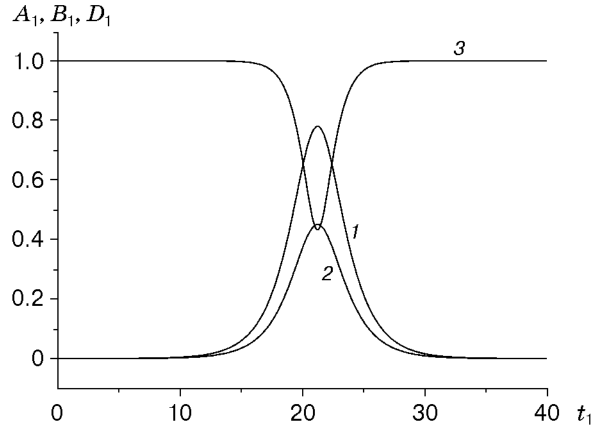

Fig. 4. Amplitudes of harmonics versus time for the separatrix phase curve $A_1^0 = B_1^0 = 0$ ($C_{00} = 1$, $C_1 = 0$, $k_1^0 = 0.5$, $k_2^0 = 0.166\,667$, and $\Delta = 1$): curves 1–3 refer to A_1, B_1, and D_1 harmonics.

As was noted above, the solution of (2.13) with $C_1 = 0$ corresponds to the separatrix $A_1^0 = 0$ and $B_1^0 = 0$; over long times, this solution is an almost periodic steady-state traveling wave with the wavenumber k_3 and amplitude D_1^0 (Fig. 4). Figure 4 shows the amplitudes of the harmonics as functions of time. Regions where all the three harmonics are substantial are seen; regions where the solution is determined by the third harmonic D_1 only are also observed. Thus, the solution is a local solitonlike structure.

For phase curves close to the separatrix $A_1^0 = 0$ and $B_1^0 = 0$, the time dependence of the harmonic amplitudes A_1 and B_1 is a train of solitons. The initial solution of (2.13) that corresponds to this case is a sinusoid with the amplitude D_1^0 during long time intervals; then, in a sufficiently narrow time interval, an "intermittency" region appears, where the amplitudes of all three harmonics are comparable. This process recurs over and over. The "intermittency" period is incomparable with the main natural periods of the H_1 wave.

Thus, for long-wave perturbations at the interface of two liquids, considered within the framework of model (1.4), the structure of wave modes with wavenumbers forming resonance triplets was examined. An analysis of the solutions of systems (2.13) and (2.16) shows that this structure is rather complex. It is shown that two families of solutions, represented in the phase space by stationary points, constitute a complex superposition of three steady-state traveling waves. In a vicinity of each of these solutions, there are families being their further complications owing to modulation of frequencies and amplitudes of the triplets in time. Under certain conditions, in the limiting case, they transform into specific solutions generally represented by a steady-state traveling wave with a constant amplitude D_1^0 and, simultaneously, in a certain narrow zone, are a superposition of three harmonics bounded by envelope solitons.

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